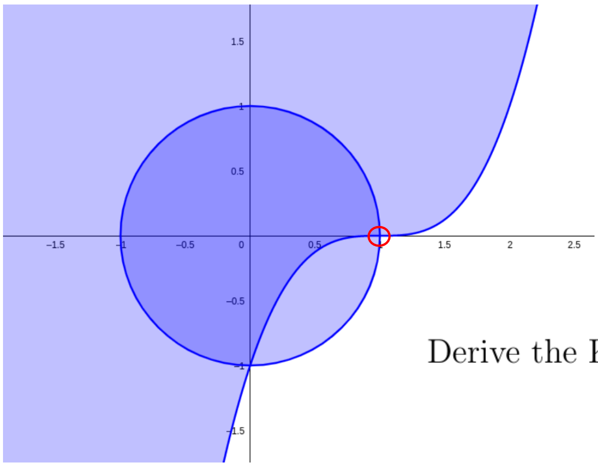
Example 8.34. Consider the problem

$$\begin{aligned} \min \quad & -x \\ s.t. \; & x^2 + y^2 \leq 1 \\ & (x-1)^3 - y \leq 0. \end{aligned}$$



Derive the KKT conditions

i) feasibility,

ii) $u_1\left(x^2+y^2-1\right)=0,\ u_1 \geq 0,$

$u_2\left((x-1)^3-y\right),\ u_2 \geq 0,$

iii) $\dfrac{\partial L}{\partial x} = -1 + 2u_1 x + 3u_2(x-1)^2 = 0,$

$\dfrac{\partial L}{\partial y} = 2u_1 y - u_2 = 0.$

pouze jeden KKT bod
(z obrázku jasné řešení)

$$(x, y, u_1, u_2) = \left(1, 0, \frac{1}{2}, 0\right).$$

Since the problem is non-convex, we can apply SOSC (8.4), (8.5). We have $I_g(1,0) = \{1,2\}$, $I_g^+(1,0) = \{1\}$ and $I_g^0(1,0) = \{2\}$. We can cumpute the gradients

$$\nabla g_1(1,0) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}|_{(1,0)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix},\ \nabla g_2(1,0) = \begin{pmatrix} 3(x-1)^2 \\ -1 \end{pmatrix}|_{(1,0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

so the conditions on $0 \neq z \in \mathbb{R}^2$ are:

$$\begin{aligned} 2z_1 &= 0, \\ -z_2 &\leq 0. \end{aligned}$$

So we have

$$Z(1,0) = \left\{z \in \mathbb{R}^2 :\ z_1 = 0,\ z_2 > 0\right\} \neq \emptyset.$$

We must compute the Heassian matrix of the Lagrange function with respect to the decision variables

$$\nabla^2_{xx} L\left(1, 0, \frac{1}{2}, 0\right) = \begin{pmatrix} 2u_1 + 6u_2(x-1) & 0 \\ 0 & 2u_1 \end{pmatrix}|_{(1,0,\frac{1}{2},0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus we have that $z^T \nabla^2_{xx} L(1,0,\frac{1}{2},0)\, z > 0$ for any $z \in Z(1,0)$, which implies that $(1,0)$ is a strict local minimum of the problem.