7 Nonlinear programming problems: Karush–Kuhn–Tucker Optimality conditions

7.1 A few pieces of the theory

We emphasize that this section contains just a basic summary and we refer the readers to the lecture notes for formal definitions and propositions.

Consider a **nonlinear programming problem** with inequality and equality constraints:

$g_i < 0 \Rightarrow u_i = 0$ | konv. opt

$g_i = 0$

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } g_i(x) \le 0,\ i = 1,\dots,m, \\ &\quad h_j(x) = 0,\ j = 1,\dots,l, \end{aligned} \tag{7.1}$$

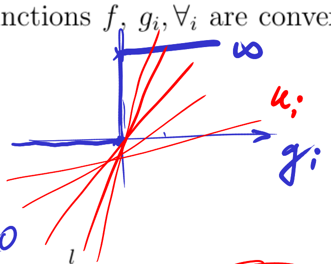
$h_j \le 0$

$h_j \ge 0$ $\quad -h_j \le 0$

where $f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}$ are differentiable functions. We denote by M the set of feasible solutions.

We say that the **problem is convex** if functions f, $g_i, \forall_i$ are convex and $h_j, \forall_j$ are affine.

$ax+b$



Define the **Lagrange function** by

≤ 0

$$L(x,u,v) = f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{j=1}^{l} v_j h_j(x),\ u_i \ge 0. \tag{7.2}$$

re

≤ 0

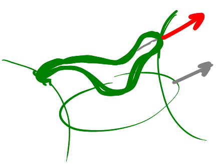
$$L(x,u,v) = f(x) \quad \max_{u_i \ge 0} + \sum_{i=1}^{m} u_i g_i(x) \quad + \sum_{j=1}^{l} v_j h_j(x)$$

$\max_{v_j, u_i \ge 0}$ $\min_x$ $f(x)$ + $u_i\ g_i(x)$

The **Karush–Kuhn–Tucker optimality conditions** are then (feasibility, complementarity and optimality):

$$\begin{aligned} &\text{přípustnost} \quad \text{i) } g_i(x) \le 0,\ i = 1,\dots,m,\ h_j(x) = 0,\ j = 1,\dots,l, \\ &\text{komplementarita} \quad \text{ii) } u_i g_i(x) = 0,\ u_i \ge 0,\ i = 1,\dots,m, \\ &\text{iii) } \boxed{\nabla_x L(x,u,v) = 0,} \end{aligned} \tag{7.3}$$

Any point (x,u,v) which fulfills the above conditions is called a KKT point.



PODMÍNKY REGULARITY

If a Constraint Qualification (CQ) condition is fulfilled, then the KKT conditions are necessary for local optimality of a point. Basic CQ conditions are:

- **Slater CQ:** $\exists \tilde{x} \in M$ such that $g_i(\tilde{x}) < 0$ for all i and the gradients $\nabla_x h_j(\tilde{x})$, $j = 1, \ldots, l$ are linearly independent.
- **Linear independence CQ** at $\hat{x} \in M$: all gradients g_i

$$\nabla_x g_i(\hat{x}),\ i \in I_g(\hat{x}),\ \nabla_x h_j(\hat{x}),\ j = 1, \ldots, l$$

  are linearly independent.

These conditions are quite strong and are sufficient for weaker CQ conditions, e.g. the Kuhn–Tucker condition (Mangasarian–Fromovitz CQ, Abadie CQ, ...).

To summarize, we are going to practice the following relations:

1. KKT point and convex problem $\rightarrow$ global optimality at x.
2. Local optimality at x and a constraint qualification (CQ) condition $\rightarrow \exists(u, v)$ such that (x, u, v) is a KKT point.

Example 7.1. Consider the nonlinear programming problem

$$\min\ (x_1-4)^2+(x_2-6)^2$$
$$\text{s.t. } x_1^2 \leq x_2, \qquad g_i \leq 0 \qquad x_1^2 - x_2 \leq 0$$
$$x_2 \leq 4.$$

Compute the Lagrange multipliers at point $\boxed{(2,4).}$ $x = (x_1, x_2) = (2,4)$

$$L(x_1, x_2, u_1, u_2) = (x_1-4)^2 + (x_2-6)^2 + u_1 \cdot (x_1^2 - x_2) + u_2(x_2-4)$$

1) $x_1^2 - x_2 \leq 0$ ✓

$x_2 - 4 \leq 0$ ✓

2) $u_1 \underbrace{(x_1^2 - x_2)}_{0} = 0$ ✓

$u_2 \underbrace{(x_2 - 4)}_{0} = 0$ ✓

$u_1 = 1$

3) $\frac{\partial L}{\partial x_1}: \quad 2(x_1 - 4) + 2x_1 u_1 = 0 \qquad -4 + 4u_1 = 0$

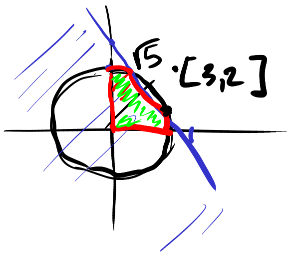
$\frac{\partial L}{\partial x_2}: \quad 2(x_2 - 6) - u_1 + u_2 = 0 \qquad -4 - u_1 + u_2 = 0$

$u_2 = 5$

$(2, 4, 1, 5)$ splňuje KKT podmínky a protože účel. fce + g_i jsou konvexní tak $(x_1, x_2) = (2,4)$ je glob. optimum

Example 7.2. (*) Consider the nonlinear programming problem

$$\begin{aligned} \min\ & (x_1-3)^2+(x_2-2)^2 \\ \text{s.t. } & x_1^2+x_2^2 \le 5, \\ & x_1+x_2 \le 3, \\ & x_1 \ge 0,\ x_2 \ge 0, \end{aligned}$$



Compute the Lagrange multipliers at point (2,1).

Example 7.3. Using the KKT conditions find the closest point to (0,0) in the set defined by

$$M = \{x \in \mathbb{R}^2 : x_1+x_2 \ge 4,\ 2x_1+x_2 \ge 5\}.$$

Can several points (solutions) exist?

$\cancel{x_1^2+x_2^2}$

$$\begin{aligned} \min\ & x_1^2+x_2^2 \\ \text{s.t. } & x_1+x_2 \ge 4 \quad u_1 \\ & 2x_1+x_2 \ge 5 \quad u_2 \end{aligned} \Bigg\}\ \text{přípustnost}$$

KKT:

$L(x_1,x_2,u_1,u_2) = x_1^2+x_2^2+u_1(4-x_1-x_2)+u_2(5-2x_1-x_2)$

2) kompl.

$u_1(\overbrace{4-x_1-x_2}^{0}) = 0$

$u_2(\overbrace{5-2x_1-x_2}^{0}) = 0$

3) $\frac{\partial L}{\partial x} = 0$:

∂L: $2x_1 - u_1 - 2u_2 = 0$

$2x_2 - u_1 - u_2 = 0$

1) obě omezení jsou aktivní:

2) $x_1+x_2 = 4$ $\quad x_1, x_2 = (1,3)$ – nejedná se o glob. opt.

3) $2x_1+x_2 = 5$

4)

$2 - u_1 - 2u_2 = 0$ $\qquad u_1 = 2-2u_2$

$6 - u_1 - u_2 = 0$ $\qquad \cancel{\ge 0}$

$6 - 2 + 2u_2 - u_2 = 0$

$4 + u_2 = 0$ $\qquad u_2 = -4$

$x_1 + x_2 = 4$ $\qquad$ $2x_1 + x_2 > 5 \Rightarrow u_2 = 0$

$$2x_1 - u_1 = 0$$
$$2x_2 - u_1 = 0$$

$x_1 = x_2 = 2$

$(x_1, x_2) = (2, 2)$

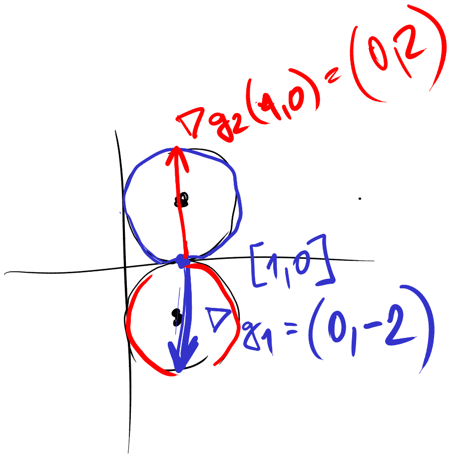
$u_1 = 4$

$(2, 2, 4, 0)$ je KKT bod

$\Rightarrow$ $(2,2)$ je glob. řeš. úlohy

Example 7.13. Consider the problem

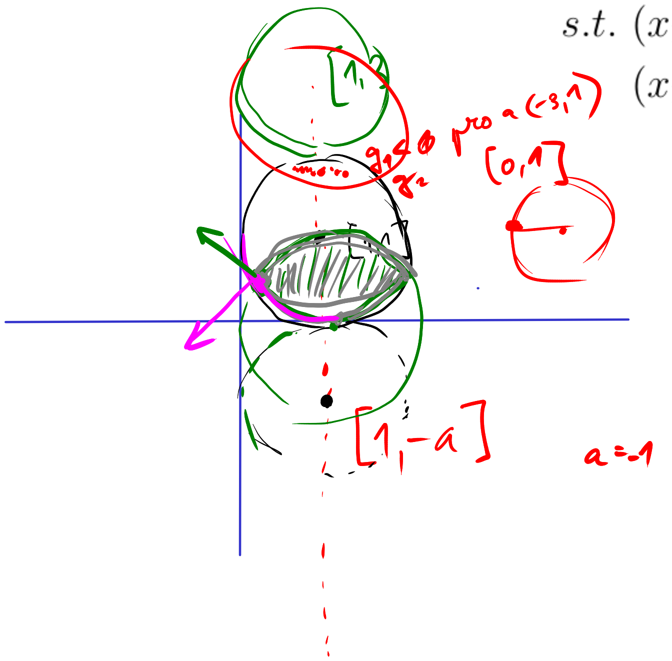
$$\min x_1$$
$$s.t.\ (x_1-1)^2+(x_2-1)^2 \le 1$$
$$(x_1-1)^2+(x_2+1)^2 \le 1.$$



Example 7.14. Consider the problem with real parameter a

$$\min x_1$$
$$s.t.\ (x_1-1)^2+(x_2-1)^2 \le 1$$
$$(x_1-1)^2+(x_2+a)^2 \le 1.$$

$g = (x-1)^2 + (x_2-1)^2 - 1 \le 0$



$a \in [-3, 1]$

že M je neprázdná